J. Fluid Mech. (1977), *vol.* 79, *part* 3, *pp.* 417–434
Printed in Great Britain

The stability of a rotating flow in the presence of an axial and a toroidal magnetic field

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(Received 21 May 1975 and in revised form 31 March 1976)

Making a normal-mode assumption, we shall investigate stability with respect to non-axisymmetric perturbations of an inhomogeneous incompressible fluid rotating between two perfectly conducting, infinite, coaxial cylinders in the presence of an axial and a toroidal magnetic field. We shall establish sufficient conditions for stability, discuss the westward-drift nature of unstable modes and estimate upper bounds on the azimuthal phase speeds and growth rates. There will also be a discussion of the validity of the sufficient conditions for stability when the normal-mode assumption is not made, so that the stability is based on an initial-value problem.

1. Introduction

Recent studies by Acheson (1972, 1973) on the stability of a uniformly rotating cylindrical flow in the presence of a magnetic field reveal that the unstable non-axisymmetric hydromagnetic waves propagate against the basic rotation. This may explain the slow, predominantly westward drift of the geomagnetic fields on the surface of the earth (cf. Acheson & Hide 1973).

Acheson's work, in which the Boussinesq approximation was made, may be summarized as follows.

(i) Unstable modes were found to drift westwards, against the basic rotation.

(ii) Conditions necessary for the excitation of special types of unstable modes (i.e. slow amplifying waves) were established.

(iii) Some upper bounds on the azimuthal phase speeds and the growth rates of the slow amplifying waves were estimated.

We shall elaborate these results.

In a cylindrical system of co-ordinates (ϖ, ϕ, z) let the azimuthal and z components of the magnetic field be denoted by B_ϕ and B_z, and those of the wavenumber by m and k respectively. The axis of rotation is chosen to coincide with the z axis. Acheson found that if $1+\beta \geqslant 0$, where $\beta \equiv \varpi k B_z/mB_\phi$, then all unstable modes drift westwards in the case of uniform rotation. In the case of differential rotation, Acheson was able to show only that they drift westwards in a frame rotating with angular velocity $\Omega_{\max}$, where $\Omega_{\max}$ is the maximum of $\Omega(\varpi)$ for all ϖ. He also observed that the westward drift is relatively insensitive to generation mechanisms such as an unstable gradient of magnetic field, rotation or density.

To study the condition necessary for the unstable modes to be excited, the axial magnetic field B_z was neglected. Because of the slow propagation and slow amplification of the geomagnetic waves, Acheson considered the modes satisfying $3\omega_r^2 \leqslant \rho m^2 \Omega_\phi^2$, where ρ is the density, Ω_ϕ the local Alfvén speed, defined as $(B_\phi^2/\rho\varpi^2)^{\frac{1}{2}}$, and ω_r the real part of the frequency with respect to a frame rotating with the uniform basic angular velocity Ω_0. Sufficient conditions for stability were then established. Some upper bounds on the azimuthal phase speeds and growth rates for these slow amplifying modes were also estimated.

In this paper, an additional non-uniform axial field $B_z(\varpi)$ will be included, and the Boussinesq approximation will not be made so that the density gradient will be properly accounted for everywhere (not just when it appears in the buoyancy term). Then we shall study the following.

(i) The westward drift of the unstable modes.

(ii) Conditions necessary for the excitation of all unstable modes, not just the slow amplifying modes.

(iii) Upper bounds on the azimuthal phase speeds and growth rates of all unstable modes.

In addition, we shall investigate the stability of some special cases when the rotation is differential. We shall also discuss the validity of sufficient conditions for stability when the normal-mode assumption is not made.

When the Boussinesq approximation is relaxed, we are unable to reproduce Acheson's conclusion that the westward drift is insensitive to an unstable density gradient ($D\rho \equiv d\rho/d\varpi > 0$). We are only able to show that if the density gradient is stable ($D\rho \leqslant 0$), and $1+\beta \geqslant 0$, then unstable modes drift westwards. Therefore, the question of whether or not the westward drift is affected by an unstable density gradient in an inhomogeneous fluid remains open.

To remove the restriction that $3\omega_r^2 \leqslant \rho m^2 \Omega_\phi^2$ so that a general sufficient condition for stability with respect to all possible disturbances can be established is of interest not only in the problem of the earth's core but also in other astrophysical and geophysical applications. For example, one of the major concerns in astrophysical problems is to construct a stellar model which is stable. Therefore the stability of all possible perturbations must be investigated. By relaxing the restriction, we establish a sufficient condition for stability which is slightly sharper than Acheson's for the $|m| = 1$ mode.

The extension (from the case of uniform rotation) to the case of differential rotation is a difficult problem. Nevertheless, Braginskiĭ [1967, equation (2.19); see also Acheson 1973, equation (7.1)] has obtained a sufficient condition for the stability of a rotating spheroid under the assumption that the deviation from (rapid) uniform rotation is very small (a more precise statement of this assumption is described by equation (59) below and a comment which follows it). This condition indicates that a rotation which increases outwards from the axis has a stabilizing influence. Since the assumption is rather strong, the validity of this stabilizing influence remains to be seen. No sufficient conditions for stability of general differential rotation have been found. However, we do obtain results in some special cases. For example, we shall consider a case where the angular velocity is assumed to be much faster than the Alfvén speed. Then we shall obtain a sufficient condition for stability which is analogous to the Richardson criterion obtained in non-magnetic plane parallel flow. This criterion sets an upper bound for a stable velocity shear.

Thermal dissipation is important in both astrophysical and geophysical problems. However, this subject will not be discussed here.

The plan of this paper is as follows. The equations of motion governing infinitesimally small perturbation are given in § 2. A brief account of the stability with respect to axisymmetric perturbations is given in § 3 mainly for the purpose of comparison with the stability results on non-axisymmetric perturbations discussed in § 4. In this section, the question of the westward drift is investigated, some sufficient conditions for stability established and some upper bounds on the azimuthal phase speeds and growth rates estimated. In § 5 it is proved that our sufficient conditions for stability in the case of uniform rotation and in the case of a strong magnetic field are valid even though the set of eigenfunctions is not complete. Finally, in § 6, we shall discuss the question of whether the sufficient conditions for stability established may be satisfied near the axis when the inner cylinder is removed.

2. The equations of motion

We consider a fluid rotating differentially between two rigid, infinite, coaxial cylinders in the presence of an axial and a toroidal magnetic field. The fluid is assumed to be incompressible but inhomogeneous and all the dissipative mechanisms such as viscosity, magnetic resistivity and thermal diffusivity are disregarded. Then the governing equations of motion in the inertial frame in cylindrical polar co-ordinates (ϖ, ϕ, z) are

$$D\mathbf{v}/Dt = -\rho^{-1}\nabla\pi + \rho^{-1}(\mathbf{B}.\nabla)\mathbf{B} - \hat{\mathbf{e}}_\varpi g_\varpi, \tag{1}$$

$$\nabla.\mathbf{v} = 0, \tag{2}$$

$$D\mathbf{B}/Dt = (\mathbf{B}.\nabla)\mathbf{v}, \quad \nabla.\mathbf{B} = 0, \tag{3}$$

where $D/Dt \equiv \partial/\partial t + \mathbf{v}.\nabla$, $\mathbf{v}$ is the velocity of the fluid, ρ the density, p the pressure, $\mathbf{B}$ the magnetic field, g_ϖ the gravitational acceleration, $\pi \equiv p + \frac{1}{2}\mathbf{B}.\mathbf{B}$ and $(\hat{\mathbf{e}}_\varpi, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_z)$ are unit vectors parallel to the three axes.

The equilibrium state is described by

$$\mathbf{v}_0 = \hat{\mathbf{e}}_\phi \varpi\Omega(\varpi), \quad \mathbf{B}_0 = \hat{\mathbf{e}}_\phi B_\phi(\varpi) + \hat{\mathbf{e}}_z B_z(\varpi), \quad \rho_0 = \rho_0(\varpi), \text{ etc.}, \tag{4}$$

which gives

$$g_\varpi - \varpi\Omega^2 = G_\varpi - \tfrac{1}{2}\varpi^2 H_\varpi - 2\varpi\Omega_\phi^2 - \rho^{-1}B_z DB_z. \tag{5}$$

The subscript zero used to indicate the equilibrium state in (4) has been dropped in (5) and will be dropped hereafter. The following notation is (or will be) used:

$$D \equiv d/d\varpi, \quad G_\varpi \equiv -\rho^{-1}Dp, \tag{6}$$

$$\Omega_\phi^2 \equiv \frac{B_\phi^2}{\rho\varpi^2}, \quad \Omega_z^2 \equiv \frac{B_z^2}{\rho\varpi^2}, \quad H_\varpi \equiv \frac{1}{\rho}D\left(\frac{B_\phi}{\varpi}\right)^2 = D\Omega_\phi^2 + \frac{1}{\rho}(D\rho)\,\Omega_\phi^2. \tag{7}$$

The boundary conditions are those of perfectly conducting rigid walls.

In linearizing the equations of motion, we shall assume that although g_ϖ depends upon the radial distance ϖ its Eulerian variation may be neglected, i.e. $\delta g = 0$. Assuming a normal-mode solution for the Lagrangian displacement $\boldsymbol{\xi} = (\xi_\varpi, \xi_\phi, \xi_z)$, i.e.

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\varpi; \phi, z)\exp(i\sigma t) = \boldsymbol{\xi}(\varpi)\exp(i\sigma t + im\phi + ikz), \tag{8}$$

and noting that the Lagrangian differential operator Δ and the Eulerian differential operator δ are related by

$$\Delta = \delta + \boldsymbol{\xi}.\nabla, \tag{9}$$

we have

$$\omega^2\boldsymbol{\xi} - 2i\omega B\boldsymbol{\xi} - C\boldsymbol{\xi} = 0, \tag{10}$$

where

$$\omega \equiv \sigma + m\Omega = \omega_r + i\omega_i, \quad iB\boldsymbol{\xi} \equiv -i\Omega(\hat{\mathbf{e}}_\varpi \xi_\phi - \hat{\mathbf{e}}_\phi \xi_\varpi), \qquad (11), (12)$$

$$C\boldsymbol{\xi} \equiv \rho^{-1}\nabla(\delta\pi) + m^2\Omega_\phi^2(1+\beta)^2\boldsymbol{\xi} + \hat{\mathbf{e}}_\varpi[N^2 + (\Phi - 4\Omega^2) - \varpi H_\varpi]\,\xi_\varpi + 2im\Omega_\phi^2(1+\beta)(\hat{\mathbf{e}}_\varpi \xi_\phi - \hat{\mathbf{e}}_\phi \xi_\varpi). \tag{13}$$

Here β is defined as

$$\beta \equiv \varpi k B_z / m B_\phi, \tag{14}$$

while the Rayleigh discriminant Φ and the Brunt–Väisälä frequency N are defined as

$$\Phi \equiv \varpi^{-3}D(\varpi^4\Omega^2), \quad N^2 \equiv -\rho^{-1}(D\rho)(g_\varpi - \varpi\Omega^2) \qquad (15), (16)$$

respectively. An explicit expression for $\delta\pi$ is not needed here and is therefore not given.

We note that in deriving (13) we have used

$$\delta\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) = imB_\phi(1+\beta)\boldsymbol{\xi} - \hat{\mathbf{e}}_\phi(DB_\phi - B_\phi/\varpi)\,\xi_\varpi - \hat{\mathbf{e}}_z(DB_z)\,\xi_\varpi, \tag{17}$$

which is obtained by linearizing (3) and by using the relation

$$\delta\mathbf{v} = i\omega\boldsymbol{\xi} - \hat{\mathbf{e}}_\phi\,\varpi D\Omega\xi_\varpi. \tag{18}$$

It is also worth pointing out that the gradient DB_z of the axial magnetic field appearing in (17) cancels out in (13).

The linearized version of the boundary condition that the radial component of velocity v_ϖ vanishes at both walls gives

$$\xi_\varpi(\varpi) = 0 \quad \text{at} \quad \varpi = \varpi_1, \varpi_2. \tag{19}$$

Thus the stability problem consists of (10) supplemented by the linearized continuity equation

$$\nabla.\boldsymbol{\xi} = D_*\,\xi_\varpi + im\varpi^{-1}\,\xi_\phi + ik\xi_z = 0, \tag{20}$$

where $D_* \equiv D + \varpi^{-1}$, and subject to boundary conditions (19).

An equation of the form of (10) is convenient for the case of uniform rotation since the frequency ω in the rotating frame is constant. For the case of differential rotation it is more convenient to rewrite (10) as

$$\sigma^2\boldsymbol{\xi} - 2i\sigma F\boldsymbol{\xi} - Q\boldsymbol{\xi} = 0, \tag{21}$$

where

$$iF\boldsymbol{\xi} \equiv -m\Omega\boldsymbol{\xi} - i\Omega(\hat{\mathbf{e}}_\varpi \xi_\phi - \hat{\mathbf{e}}_\phi \xi_\varpi), \tag{22}$$

$$Q\boldsymbol{\xi} = -m^2\Omega^2\boldsymbol{\xi} - 2im\Omega^2(\hat{\mathbf{e}}_\varpi \xi_\phi - \hat{\mathbf{e}}_\phi \xi_\varpi) + C\boldsymbol{\xi}. \tag{23}$$

The operator C is defined by (13).

Another alternative form of (10), or (21), is also of interest. Eliminating ξ_ϕ between the ϖ and the z component of (10) and also eliminating ξ_ϕ and ξ_z among the ϕ and z components of (10) and (20), we obtain

$$\left\{\omega^2 - m^2\Omega_\phi^2(1+\beta)^2 - (N^2 + 2\varpi\Omega D\Omega - \varpi H_\varpi) - \frac{4[\omega\Omega - m\Omega_\phi^2(1+\beta)]^2}{\omega^2 - m^2\Omega_\phi^2(1+\beta)^2}\right\}\xi_\varpi = \frac{1}{\rho}D(\delta\pi) + \frac{2m[\omega\Omega - m\Omega_\phi^2(1+\beta)]}{\varpi[\omega^2 - m^2\Omega_\phi^2(1+\beta)^2]}\frac{\delta\pi}{\rho} \tag{24}$$

and

$$[\omega^2 - m^2\Omega_\phi^2(1+\beta)^2]D_*\xi_\varpi - 2m\varpi^{-1}[\omega\Omega - m\Omega_\phi^2(1+\beta)]\xi_\varpi = M^2\delta\pi/\rho \tag{25}$$

respectively, where we define

$$M^2 \equiv k^2 + m^2/\varpi^2. \tag{26}$$

Finally, eliminating $\delta\pi$ between (24) and (25) gives

$$D\left\{\frac{\rho}{M^2}[\omega^2 - m^2\Omega_\phi^2(1+\beta)^2]D_*\xi_\varpi\right\} - \varpi D\left\{\frac{\rho}{M^2}[\omega\Omega - m\Omega_\phi^2(1+\beta)]\left(\frac{2m}{\varpi^2}\right)\right\}\xi_\varpi + \rho[-\omega^2 + N^2 + 2\varpi\Omega D\Omega - \varpi H_\varpi + m^2\Omega_\phi^2(1+\beta)^2]\xi_\varpi + \frac{[\omega\Omega - m\Omega_\phi^2(1+\beta)]^2}{\omega^2 - m^2\Omega_\phi^2(1+\beta)^2}\left(\frac{4\rho k^2}{M^2}\right)\xi_\varpi = 0. \tag{27}$$

All the equations above are formulated in the inertial frame. It is sometimes more convenient to consider motion in a uniformly rotating frame, for example when the upper bound of the azimuthal phase speeds is to be estimated (cf. §4.3) and when sufficient conditions for stability are to be established (cf. case (iii) in §4.2). We decompose the rotation $\Omega(\varpi)$ into a uniform part Ω_0 and a differential part $\Omega_1(\varpi)$:

$$\Omega(\varpi) = \Omega_0 + \Omega_1(\varpi). \tag{28}$$

The uniform rotation Ω_0 is left unspecified here but will be specified later [e.g. (64)]. Then in a frame rotating with Ω_0 (this frame will be called the Ω_0 frame hereafter), the linearized equation of motion written in terms of the Lagrangian displacement

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\varpi)\exp(iqt + im\phi + ikz) \tag{29}$$

is

$$q^2\boldsymbol{\xi} - 2iqK\boldsymbol{\xi} - H\boldsymbol{\xi} = 0, \tag{30}$$

where

$$K\boldsymbol{\xi} = im\Omega_1\boldsymbol{\xi} - (\Omega_0 + \Omega_1)(\hat{\mathbf{e}}_\varpi\xi_\phi - \hat{\mathbf{e}}_\phi\xi_\varpi), \tag{31}$$

$$H\boldsymbol{\xi} = -m^2\Omega_1^2\boldsymbol{\xi} - 2im[\Omega_1(\Omega_0 + \Omega_1) - \Omega_\phi^2(1+\beta)](\hat{\mathbf{e}}_\varpi\xi_\phi - \hat{\mathbf{e}}_\phi\xi_\varpi) + \rho^{-1}\nabla(\delta\pi) + m^2\Omega_\phi^2(1+\beta)^2\boldsymbol{\xi} + \hat{\mathbf{e}}_\varpi[N^2 + 2\varpi(\Omega_0 + \Omega_1)D\Omega_1 - \varpi H_\varpi]\xi_\varpi. \tag{32}$$

The notation here is the same as before. We note that (30) supplemented by (31) and (32) reduces to (21) supplemented by (22) and (23) if we set $\Omega_0 \equiv 0$ and replace q and Ω_1 by σ and Ω.

It is important to point out that all the operators involved, i.e. iB, C, iF, Q, iK and H, are Hermitian. This property is crucial to our stability analysis. We shall prove this property later by showing that the proper inner products have real values.

We define an inner product for any operator A as

$$(\boldsymbol{\xi}, A\boldsymbol{\xi}) \equiv \int_R \rho\boldsymbol{\xi}^* . A\boldsymbol{\xi}\, d\mathbf{x} \equiv \int \rho\boldsymbol{\xi}^* . A\boldsymbol{\xi}, \tag{33}$$

where R is the region occupied by the fluid in the equilibrium state, $\boldsymbol{\xi}^*$ is the complex conjugate of ξ and the last identity is introduced as shorthand. For example, if the equilibrium configuration is cylindrical, R is taken as the volume between the two coaxial cylinders bounded by a proper length in the z direction. For simplicity, we shall write

$$\int_{\varpi_1}^{\varpi_2} d\varpi\varpi\rho\boldsymbol{\xi}^* . A\boldsymbol{\xi} \equiv (\boldsymbol{\xi}, A\boldsymbol{\xi}). \tag{34}$$

With the above definition of an inner product, the frequency of a small perturbation follows from, say, (21) as

$$\sigma = \frac{(\boldsymbol{\xi}, iF\boldsymbol{\xi}) \pm [(\boldsymbol{\xi}, iF\boldsymbol{\xi})^2 + (\boldsymbol{\xi}, \boldsymbol{\xi})(\boldsymbol{\xi}, Q\boldsymbol{\xi})]^{\frac{1}{2}}}{(\boldsymbol{\xi}, \boldsymbol{\xi})}. \tag{35}$$

Suppose that iF and Q are both Hermitian operators. Then the stability is determined by the sign of the real discriminant

$$D \equiv (\boldsymbol{\xi}, iF\boldsymbol{\xi})^2 + (\boldsymbol{\xi}, \boldsymbol{\xi})(\boldsymbol{\xi}, Q\boldsymbol{\xi}). \tag{36}$$

For unstable modes (i.e. $D < 0$), the real and imaginary parts of σ are given by

$$\sigma_r = (\boldsymbol{\xi}, iF\boldsymbol{\xi})/(\boldsymbol{\xi}, \boldsymbol{\xi}) \tag{37}$$

and

$$\sigma_i = (-D)^{\frac{1}{2}}/(\boldsymbol{\xi}, \boldsymbol{\xi}) \tag{38}$$

respectively.

3. Stability of axisymmetric perturbations ($m = 0$)

The criteria for stability with respect to axisymmetric perturbations are relatively well understood and are associated with Rayleigh for the non-magnetic case and with Michael for the magnetic case when $B_\phi \neq 0$ but $B_z = 0$. We shall briefly discuss them for the purposes of comparison with those for non-axisymmetric perturbations. Readers are referred to Chandrasekhar (1961), Howard & Gupta (1962) and Roberts & Soward (1972) for further details.

The case $B_z \neq 0$ is of particular interest for two reasons. One is that, as B_z approaches zero, the stability criterion (45) obtained with $\Omega \neq 0$ and $B_\phi \neq 0 \neq B_z$ does not approach the Michael criterion (42), nor the Rayleigh criterion when we further set $B_\phi = 0$. This is a well-known stability paradox. The other reason is the loss of the conservation of the angular momentum per unit mass in the z direction (denoted by h hereafter) even though the perturbations are axisymmetric.

For the removal of the paradox, readers are referred to the literature available (see, for example, Velikhov (1959) and Chandrasekhar (1961, p. 389) for removal by means of magnetic dissipation and Howard & Gupta (1962) for removal by an ingenious argument without invoking dissipation).

The Lagrangian variation in h is (cf. Lynden-Bell & Ostriker 1967, p. 304)

$$\Delta h \equiv \Delta(\varpi \hat{\mathbf{e}}_\phi . \mathbf{v}) = i\varpi(\sigma\xi_\phi - 2i\Omega\xi_\varpi). \tag{39}$$

Thus, if h is conserved (i.e. $\Delta h = 0$), we have

$$\sigma\xi_\phi - 2i\Omega\xi_\varpi = 0 \tag{40}$$

and vice versa. An immediate consequence of $\Delta h \neq 0$ in the non-magnetic case is the loss of some (or perhaps all) of the stabilizing effect of rotation which exists when $\Delta h = 0$. Therefore, in the absence of density stratification one may expect rotational shear instabilities to arise in some cases even though $\Phi > 0$. In the presence of a magnetic field, the ϕ component of (21) gives, for $m = 0$,

$$\sigma(\sigma\xi_\phi - 2i\Omega\xi_\varpi) = (kB_z/\varpi\rho)(\varpi kB_z\xi_\phi - 2iB_\phi\xi_\varpi). \tag{41}$$

Clearly, $\Delta h \neq 0$ if $B_z \neq 0$ (we exclude here special perturbations satisfying $\varpi kB_z\xi_\phi - 2iB_\phi\xi_\varpi = 0$ and assume that $k \neq 0$).

In the absence of an axial field, relation (40) is valid and we may readily obtain the necessary and sufficient condition for stability as

$$N^2 + \Phi - \varpi H_\varpi' \geqslant 0 \quad \text{everywhere.} \tag{42}$$

This criterion was first derived by Michael for ρ = constant (see Roberts & Soward (1972, equation 2.46) for an extension to include the buoyant force) and will be called the Michael criterion. We note that it reduces to the Rayleigh criterion when $B_\phi = 0$. When $B_z \neq 0$, (40) is not valid and we must return to (21)–(23). We obtain

$$(\boldsymbol{\xi}, iF\boldsymbol{\xi}) = -i\int \rho\Omega(\xi_\varpi^*\xi_\phi - \xi_\varpi\xi_\phi^*), \tag{43}$$

$$(\boldsymbol{\xi}, Q\boldsymbol{\xi}) = \int \rho\left[\Omega_\phi^2\left|2\xi_\varpi + \frac{i\varpi kB_z}{B_\phi}\xi_\phi\right|^2 + \varpi^2k^2\Omega_z^2|\xi_z|^2 + (N^2 + 2\varpi\Omega D\Omega - \varpi H_\varpi - 4\Omega_\phi^2 + \varpi^2k^2\Omega_z^2)|\xi_\varpi|^2\right]. \tag{44}$$

The Hermitian property of iF and Q is readily observed by noting that both inner products are real. Then a sufficient condition for stability follows from (44) as

$$N^2 + 2\varpi\Omega D\Omega - \varpi H_\varpi - 4\Omega_\phi^2 + \varpi^2k^2\Omega_z^2 \geqslant 0 \quad \text{everywhere.} \tag{45}$$

This condition was first derived by Howard & Gupta for ρ and B_z constant and $g_\varpi = 0$ and will be called the Howard–Gupta criterion. We note that this criterion is not altered when B_z is non-uniform. This is because the term involving the gradient of B_z has cancelled out in the derivation (13) as has already been mentioned. This cancellation is accidental and due to the geometry of an infinite cylinder. For a spheroidal configuration, the gradient of B_z may appear.

Perhaps the most important effect of B_z on stability is the loss of the conservation of angular momentum h. As a consequence, 'shear instabilities', both rotational and magnetic, become possible in some cases. This may be inferred from a comparison of (42) and (45). We observe that the stabilizing term

$4(\Omega^2+\Omega_\phi^2)$ appearing in (42), where $B_z = 0$, is lost in (45), where $B_z \neq 0$. Certainly, this inference cannot be regarded as final since (45) is only a sufficient condition for stability. Nevertheless, a physical argument for stability based on the loss of the conservation of angular momentum per unit mass tends to support the conjecture made here.

4. Stability of non-axisymmetric perturbations ($m \neq 0$)

In the following, we shall investigate the effect of the density stratification on the various results obtained by Acheson under the Boussinesq approximation.

4.1. *Westward drift*

We shall define some terminology. For the linearized equations of motion formulated in the inertial frame, the angular velocity of wave propagation is given by $\dot\phi_I \equiv -\sigma_r/m$. The angular velocity of wave propagation with respect to a frame rotating with a local angular velocity $\Omega(\varpi)$ is then given by

$$\dot\phi_R \equiv \dot\phi_I - \Omega(\varpi) = -\omega_r/m.$$

Thus, if $\dot\phi_R = -\omega_r/m < 0$, then the unstable modes propagate westwards with respect to a frame rotating with a local angular velocity $\Omega(\varpi)$.

Multiplying (27) by $\varpi\xi_\varpi^*$ and then integrating over (ϖ_1, ϖ_2) gives

$$\int \rho\left\{\frac{1}{M^2}[\omega^2-m^2\Omega_\phi^2(1+\beta)^2]\,|D_*\,\xi_\varpi|^2-\frac{1}{M^2}\left[\omega\Omega-m\Omega_\phi^2(1+\beta)\left(\frac{2m}{\varpi}\right)\left(D+\frac{2}{\varpi}\right)\right]|\xi_\varpi|^2\right.$$
$$+[\omega^2-m^2\Omega_\phi^2(1+\beta)^2-(N^2+2\varpi\Omega D\Omega-\varpi H_\varpi)]\,|\xi_\varpi|^2$$
$$\left.-\frac{[\omega\Omega-m\Omega_\phi^2(1+\beta)]^2\,4k^2}{\omega^2-m^2\Omega_\phi^2(1+\beta)^2\,M^2}|\xi_\varpi|^2\right\}=0. \qquad (46)$$

The imaginary part of (46) is

$$(2i\omega_i)\int\frac{\rho\omega_r}{M^2}\left\{|D\xi_\varpi|^2+\left[\frac{1}{\varpi}\left(-\frac{D\rho}{\rho}\right)+\frac{1}{M^2}\left(\frac{m^2(m^2-1)}{\varpi^4}+k^2(k^2+\frac{2m^2}{\varpi^2}+\frac{1}{\varpi^2}\right)\right.\right.$$
$$\left.+\frac{4k^2m^2\Omega_\phi^2(1+\beta)^2\,(\Omega^2+\Omega_\phi^2)}{|\omega^2-m^2\Omega_\phi^2(1+\beta)^2|^2}\right]|\xi_\varpi|^2$$
$$\left.-\left(\frac{m\Omega}{\omega_r}\right)\left[\frac{2k^2}{\varpi^2M^2}+\frac{1}{\varpi}\left(-\frac{D\rho}{\rho}\right)+\frac{4k^2[|\omega|^2+m^2\Omega_\phi^2(1+\beta)^2]\,\Omega_\phi^2(1+\beta)}{|\omega^2-m^2\Omega_\phi^2(1+\beta)^2|^2}\right]|\xi_\varpi|^2\right\}=0. \qquad (47)$$

We note that for unstable modes ($\omega_i \neq 0$) the integrand in (47) must vanish at least somewhere.

In order to prove the westward drift, we need a lemma which ensures that unstable modes must propagate (provided (49) and (50) are satisfied). That is, if $\omega_i \neq 0$ then $\omega_r \neq 0$. Acheson proved this lemma by an argument of contradiction. Suppose that $\omega_r = 0$ when $\omega_i \neq 0$; then (47) reduces to

$$(2i\omega_i)\int\frac{\rho m\Omega}{M^2}\left\{\frac{2k^2}{\varpi^2M^2}+\frac{1}{\varpi}\left(-\frac{D\rho}{\rho}\right)+\frac{4k^2[|\omega|^2+m^2\Omega_\phi^2(1+\beta)^2]\,\Omega_\phi^2(1+\beta)}{|\omega^2-m^2\Omega_\phi^2(1+\beta)^2|^2}\right\}|\xi_\varpi|^2=0. \qquad (48)$$

Thus, for $\omega_i \neq 0$ the integrand must vanish somewhere. This is not possible for $k \neq 0$ if neither

$$1+\beta \geqslant 0 \tag{49}$$

nor

$$-D\rho \geqslant 0 \tag{50}$$

is violated. We thus have a contradiction if (49) and (50) are satisfied simultaneously. The westward drift follows from (47) provided that both (49) and (50) are satisfied. This can be seen in the following manner. In the case of uniform rotation, ω_r is constant and the vanishing of the integrand in (47) requires that $-m\Omega/\omega_r < 0$. This implies that $-\omega_r/m < 0$ since Ω is chosen as positive. In the case of differential rotation, we require $-m\Omega(\varpi)/\omega_r(\varpi) < 0$ for, at least, some ϖ in (ϖ_1, ϖ_2). This is the same as requiring that $-\omega_r/m = \dot{\phi}_I - \Omega(\varpi) < 0$ for some ϖ. Therefore we have $\dot{\phi}_I < \Omega(\varpi) \leqslant \Omega_{\max}$, where $\Omega_{\max} \equiv \sup_{\varpi} \Omega(\varpi)$, and all unstable modes propagate westwards with respect to a frame rotating with $\Omega_{\max}$. Thus Acheson's results are recovered.

When the Boussinesq approximation is made, density stratification appears in the real part of (46) but not in its imaginary part [i.e. (47)]. Acheson, therefore, noted that the westward drift is not affected by a top-heavy density gradient ($D\rho > 0$). Whether or not this conclusion remains valid when the Boussinesq approximation is not made remains an open question. We note that, in the method used here, because of the presence of the density-stratification terms in the imaginary part (47) we are able to prove neither the lemma that if $\omega_i \neq 0$ then $\omega_r \neq 0$ nor the westward drift of the unstable modes when $D\rho > 0$.

The effect of magnetic-field distribution on the westward drift has been discussed by Acheson (1973) while the effect of a spheroidal equilibrium configuration, instead of a cylindrical one, has been discussed, for example, by Hide (1966) and by Malkus (1967). The question of why the unstable modes must drift westwards has yet to be answered. Noting that this tendency also prevails in the absence of a magnetic field, perhaps an explanation should be sought in the simpler problem of the non-magnetic case as a first step.

4.2. *Sufficient conditions for stability*

In the following, we shall establish sufficient conditions for stability when $B_\phi \neq 0$ and $B_z = 0$ and in some cases when $B_\phi \neq 0 \neq B_z$. In the case of general differential rotation, we are unable to obtain a satisfactory sufficient condition for stability. Nevertheless, we shall briefly discuss the general case to demonstrate an approach which may be improved to obtain a more satisfactory result. For some special cases we are able to obtain sufficient conditions for stability of some practical importance. The special cases of slight differential rotation and strong magnetic field, in a spheroidal configuration, have been discussed by Braginskiĭ (1967) and by Roberts & Soward (1972) respectively. We shall rediscuss them in the context of a cylindrical flow with their limitations more precisely stated. These limitations are important restrictions and could be rather severe in some cases. We shall also discuss the case of rapid rotation.

(i) *General differential rotation.* For unstable modes ($\omega_i \neq 0$) we may introduce a change of variables

$$\xi_\varpi = \omega^{-\frac{1}{2}}\eta. \tag{51}$$

Then (27) becomes

$$D\left\{\frac{\rho}{M^2}\left[\omega-\frac{1}{\omega}m^2\Omega_\phi^2(1+\beta)^2\right]D_*\eta\right\}-m\varpi D\left\{\frac{\rho}{\varpi^2M^2}\left[2\Omega+\tfrac{1}{2}\varpi D\Omega-\frac{2}{\omega}m\Omega_\phi^2(1+\beta)\right.\right.$$
$$\left.\left.-\frac{\varpi}{2\omega^2}m^2\Omega_\phi^2(1+\beta)^2(D\Omega)\right]\right\}\eta$$
$$+\frac{1}{\omega^2}\frac{2\rho m D\Omega}{\varpi M^2}m^2\Omega_\phi^2(1+\beta)\eta+\frac{1}{\omega^3}\frac{\rho(D\Omega)^2}{4M^2}m^4\Omega_\phi^2(1+\beta)^2\eta$$
$$+\frac{[\omega\Omega-m\Omega_\phi^2(1+\beta)]^2}{\omega[\omega^2-m^2\Omega_\phi^2(1+\beta)^2]}\frac{4\rho k^2}{M^2}\eta$$
$$+\frac{\rho}{\omega}\left[-\omega^2+N^2+\frac{k^2}{M^2}(2\varpi\Omega D\Omega)-\frac{m^2(D\Omega)^2}{4M^2}-\varpi H_\varpi+m^2\Omega_\phi^2(1+\beta)^2\right]\eta=0. \tag{52}$$

We obtain an integral by first multiplying (52) by $\varpi\eta^*$ and then integrating over (ϖ_1, ϖ_2). The imaginary part of this integral can be written in the form

$$(i\omega_i)\int\rho A = 0, \tag{53}$$

where A is given by

$$A \equiv \frac{1}{M^2}|D_*\eta|^2+|\eta|^2+\frac{m^2\Omega_\phi^2}{|\omega|^2M^2}\left\{\left[|1+\beta||D_*\eta|-\left|\frac{2}{\varpi}+\frac{\omega_r(1+\beta)mD\Omega}{|\omega|^2}\right||\eta|\right]^2\right.$$
$$+2|1+\beta|\left|\frac{2}{\varpi}+\frac{\omega_r(1+\beta)mD\Omega}{|\omega|^2}\right||D_*\eta||\eta|-(1+\beta)\left[\frac{2}{\varpi}+\frac{\omega_r(1+\beta)mD\Omega}{|\omega|^2}\right]$$
$$\left.\times\left(D+\frac{2}{\varpi}\right)|\eta|^2\right\}$$
$$+\frac{1}{|\omega^2-m^2\Omega_\phi^2(1+\beta)^2|^2}\{[|\omega|^2+m^2\Omega_\phi^2(1+\beta)^2](\Omega^2+\Omega_\phi^2)$$
$$-4\omega_r m\Omega\Omega_\phi^2(1+\beta)\}\frac{4k^2}{M^2}|\eta|^2+|\omega|^{-2}T|\eta|^2. \tag{54}$$

It can be shown that the first three lines of (54) are all positive. Thus we obtain as a sufficient condition for stability

$$T \equiv N^2+\frac{k^2}{M^2}(2\varpi\Omega D\Omega)-\frac{m^2}{M^2}\frac{(D\Omega)^2}{4}-\varpi H_\varpi-4\Omega_\phi^2$$
$$+m^2\Omega_\phi^2(1+\beta)^2\left[1-\frac{m^2(D\Omega)^2}{4|\omega|^2M^2}\right] \geqslant 0 \quad \text{everywhere.} \tag{55}$$

Two limiting cases are of interest. When the magnetic field is absent, (55) reduces to

$$N^2/\varpi^2(D\Omega)^2 \geqslant \tfrac{1}{4}, \tag{56}$$

where we have used

$$0 \leqslant k^2/M^2, \quad m^2/\varpi^2M^2 \leqslant 1. \tag{57}$$

This is a modified form of the Richardson criterion so familiar in plane parallel flow (cf. Howard & Gupta 1962). On the other hand, when the rotation is uniform, (55) gives

$$N^2 - \varpi H_\varpi - 4\Omega_\phi^2 + m^2\Omega_\phi^2(1+\beta)^2 \geqslant 0. \tag{58}$$

This is a generalized and somewhat improved version of the sufficient condition for stability first established by Acheson [1973, equation (5.5)] for $B_z = 0$ and under the Boussinesq approximation. We note that this sufficient condition for stability may be obtained more directly by taking inner products of (10). Then the condition for positive definiteness of $(\boldsymbol{\xi}, C\boldsymbol{\xi})$ gives the sufficient condition for stability (58).

The applicability of the sufficient condition (55) is greatly restricted by the appearance of $|\omega|^2$ in the term coupling the differential rotation and magnetic field. However, in certain limited circumstances it may still be of some use. For example, if we are only interested in oscillation periods within a certain range, then the lower bound of this range gives a lower bound on ω_r^2 which in turn can be substituted for $|\omega|^2$. Then (55) can be easily applied to check stability.

(ii) *Slight differential rotation.* The case when the deviation from uniform rotation is very small has been investigated by Braginskiĭ (1967) for a spheroidal configuration. We shall apply this assumption to differentially rotating flow between two coaxial cylinders but with the conditions required for such an assumption more precisely stated. It will be found that the conditions required are rather severe. Consequently, the significance and, even, the validity of the sufficient conditions derived remain to be demonstrated.

The linearized equation of motion is (21) with iF and Q defined by (22) and (23) respectively. We decompose the rotation into a constant and a differential part as in (28). In assuming slight differential rotation, we require that

$$|\Omega_1/\Omega_0| \ll 1, \quad |\varpi D\Omega_1/\Omega_1| \gg 1 \quad \text{for all} \quad \varpi. \tag{59a,b}$$

Restriction (59a) is a reasonable one since it only requires that the rotation deviates slightly from a uniform one. For example, in the problem of the earth's core dynamo (cf. Braginskiĭ 1967, p. 852), we have the following relative orders of magnitude for the basic uniform rotation, the local Alfvén speed due to B_ϕ and the deviation from a uniform rotation:

$$2\Omega_0 : \Omega_\phi : \Omega_1 \sim 10^6 : 10^3 : 1. \tag{60}$$

We note that (59a) is reasonably satisfied. Restriction (59b) is a very severe one which requires that the gradient of deviation at any point must be very large in comparison with the deviation itself at that point. This can be true only when the ratio Ω_1/Ω_0 is so small that it will remain so everywhere even for large $|D\Omega_1|$.

Requirement (59a) assures that Ω_1 is negligible in comparison with Ω_0. Requirement (59b) justifies retention of $2\varpi\Omega_0 D\Omega$, in comparison with $2m^2\Omega_0\Omega_1$, which appears in the expansion of $m^2\Omega^2 = m^2(\Omega_0 + \Omega_1)^2$ [cf. (23) and (13)]. One may argue that $2m^2\Omega_0\Omega_1$ may be neglected in comparison with $m^2\Omega_\phi^2(1+\beta)^2$ without invoking (59b). But then, to be consistent, the term $2\varpi\Omega_0 D\Omega_1$ must be neglected as well. Then the stabilizing influence of rotation when $D\Omega_1 > 0$ does not appear in the expression for Q in (23).

Applying assumptions (59), we have as the simplified forms of (22) and (23)

$$iF\boldsymbol{\xi} \sim -m\Omega_0\boldsymbol{\xi} - i\Omega_0(\hat{\mathbf{e}}_\varpi \xi_\phi - \hat{\mathbf{e}}_\phi \xi_\varpi), \tag{61}$$

$$Q\boldsymbol{\xi} \sim -m^2\Omega_0^2\boldsymbol{\xi} - 2im[\Omega_0^2 - \Omega_\phi^2(1+\beta)](\hat{\mathbf{e}}_\varpi \xi_\phi - \hat{\mathbf{e}}_\phi \xi_\varpi) + \rho^{-1}\nabla(\delta\pi) + m^2\Omega_\phi^2(1+\beta)^2\boldsymbol{\xi} + \hat{\mathbf{e}}_\varpi(N^2 + 2\varpi\Omega_0 D\Omega_1 - \varpi H_\varpi)\xi_\varpi. \tag{62}$$

Then the stability discriminant D defined by (36) is readily computed and a sufficient condition for stability follows as

$$N^2 + 2\varpi\Omega_0 D\Omega_1 - \varpi H_\varpi - 4\Omega_\phi^2 + m^2\Omega_\phi^2(1+\beta)^2 \geqslant 0 \quad \text{everywhere.} \tag{63}$$

We note that the rotation has a stabilizing influence if it increases outwards from the axis. We also note that if $\Omega_1 = 0$ then (63) reduces to the sufficient condition for stability (58) for the case of uniform rotation.

(iii) *Strong magnetic field* ($\Omega_\phi^2 \geqslant \Omega_1^2$). The title 'strong magnetic field' is somewhat misleading and we shall give a more precise description. Decomposing the rotation in the manner of (28), we define the uniform part as

$$\Omega_0 \equiv \min_\varpi \Omega(\varpi). \tag{64}$$

Clearly, we have $\Omega_1(\varpi) \geqslant 0$ for all ϖ. Then 'strong magnetic field' means the case with the property that

$$\Omega_\phi^2 \geqslant \Omega_1^2. \tag{65}$$

This does not necessarily mean that the magnetic energy is required to be larger than the rotational energy since it is permitted that Ω_0^2 be much larger than Ω_ϕ^2. The restriction required here is much less severe than that in the case of slight differential rotation since (59) is not needed and (65) can be satisfied in problems such as the problem of the earth's core [cf. (60)]. We shall see shortly, however, that in order to obtain stability a much stronger magnetic field than (65) is required [cf. (68)]. This perhaps justifies somewhat the title 'strong magnetic field'.

Considering motion in the Ω_0 frame and putting $B_z = 0$, we readily obtain from (31) and (32)

$$(\boldsymbol{\xi}, iK\boldsymbol{\xi}) = -m\int \rho\Omega_1|\xi|^2 - i\int \rho(\Omega_0+\Omega_1)(\xi_\varpi^*\xi_\phi - \xi_\varpi\xi_\phi^*), \tag{66}$$

$$(\boldsymbol{\xi}, H\boldsymbol{\xi}) = \int \rho[m^2\Omega_0\Omega_1|\xi_\phi|^2 + m^2(\Omega_\phi^2 - \Omega_1^2)|\xi_z|^2 + T_0|2\xi_\varpi + im\xi_\phi|^2 + T|\xi_\varpi|^2], \tag{67}$$

where T_0 and T are given by (68) and (69) respectively. We note that both ($\boldsymbol{\xi}$, $iK\boldsymbol{\xi}$) and ($\boldsymbol{\xi}$, $H\boldsymbol{\xi}$) are real. Thus iK and H are Hermitian. Then a sufficient condition for stability follows from (67):

$$T_0 \equiv \Omega_\phi^2 - \Omega_1(\Omega_0 + \Omega_1) \geqslant 0 \tag{68}$$

and

$$T \equiv N^2 + 2(\Omega_0+\Omega_1)(\varpi D\Omega_1 + 2\Omega_1) - \varpi H_\varpi - 4\Omega_\phi^2 + m^2(\Omega_\phi^2 - \Omega_1^2) \geqslant 0 \tag{69}$$

must be satisfied simultaneously everywhere. Here we have assumed that (65) holds. A sufficient condition for stability of this type has been obtained by Roberts & Soward (1972) for a spheroidal configuration. As noted earlier, a much stronger magnetic field is required by (68) than that by (65) in order to obtain stability. However, (68) can be satisfied, for example, in the problem of the earth's core [cf. (60)]. We have not been able to obtain a meaningful sufficient condition for stability when $B_z \neq 0$.

(iv) *Rapid rotation.* In the earth's core, the rotation is much faster than the local Alfvén speed but the gradient of rotation may not be as steep as the gradient of the magnetic field. For problems of this type, the following conditions may be satisfied:

$$\Omega^2 \gg \Omega_\phi^2 \quad \text{but} \quad |D\Omega^2| \sim |D\Omega_\phi^2|. \tag{70}$$

Assuming (70) and $B_z = 0$, the terms $m^2\Omega_\phi^2$ and $m\Omega_\phi^2$ may be neglected in comparison with $m^2\Omega^2$ and $m\Omega^2$ respectively in (27). Then (27) simplifies to

$$D\left[\omega^2\frac{\rho}{M^2}D_*\,\xi_\varpi\right] - \varpi D\left[\omega\frac{2\rho m\Omega}{\varpi^2M^2}\right]\xi_\varpi + \rho\left[-\omega^2+N^2+2\varpi\Omega D\Omega+\frac{4k^2\Omega^2}{M^2}-\frac{2B_\phi DB_\phi}{\rho\varpi}\right]\xi_\varpi = 0. \tag{71}$$

Following the same procedure as in the case of general differential rotation, we introduce the change of variable (51) to reduce (71) to

$$D\left[\omega\frac{\rho}{M^2}D_*\,\eta\right] - m\varpi D\left[\frac{\rho(2\Omega+\frac{1}{2}\varpi D\Omega)}{\varpi^2M^2}\right]\eta - \frac{\rho}{\omega}\left\{\omega^2-\left[N^2+\frac{k^2\Phi}{M^2}-\frac{1}{4}\frac{m^2(D\Omega)^2}{M^2}-\frac{2B_\phi DB_\phi}{\rho\varpi}\right]\right\}\eta = 0. \tag{72}$$

Multiplying (72) by $\omega\eta^*$ and integrating over (ϖ_1, ϖ_2) gives an integral whose imaginary part can be written in the form

$$(i\omega_i)\int \rho A = 0, \tag{73}$$

with A given by

$$A \equiv \frac{1}{M^2}|D_*\,\eta|^2+|\eta|^2+\left[N^2+\frac{k^2\Phi}{M^2}-\frac{m^2(D\Omega)^2}{4M^2}-\frac{2B_\phi DB_\phi}{\rho\varpi}\right]\left|\frac{\eta}{\omega}\right|^2. \tag{74}$$

Thus a sufficient condition for stability is

$$N^2+\frac{k^2\Phi}{M^2}-\frac{m^2(D\Omega)^2}{4M^2}-\frac{2B_\phi DB_\phi}{\rho\varpi} \geqslant 0 \quad \text{everywhere.} \tag{75}$$

A more convenient alternative form is

$$\frac{N^2-2B_\phi DB_\phi/\rho\varpi}{\varpi^2(D\Omega)^2} \geqslant \tfrac{1}{4}. \tag{76}$$

We note a similarity between (76) and the modified Richardson criterion (56) for a rotating flow.

4.3. *Upper bounds on the azimuthal phase speeds and growth rates*

If we consider motion in the Ω_0 frame, where Ω_0 is defined by (64), then the equation of motion is (30) with the inner products of iK and H given by (66) and (67). An alternative form of (67) is

$$(\boldsymbol{\xi}, H\boldsymbol{\xi}) = \int \rho\{-m^2\Omega_1^2|\boldsymbol{\xi}|^2 - 2im\Omega_1(\Omega_0+\Omega_1)(\xi_\varpi^*\xi_\phi - \xi_\varpi\xi_\phi^*) + m^2\Omega_\phi^2(1+\beta)^2|\xi_z|^2 + \Omega_\phi^2|2\xi_\varpi + im(1+\beta)\xi_\phi|^2 + [N^2 + 2\varpi(\Omega_0+\Omega_1)D\Omega_1 - \varpi H_\varpi - 4\Omega_\phi^2 + m^2\Omega_\phi^2(1+\beta)^2]|\xi_\varpi|^2\}. \quad (77)$$

Because of the Hermitian property of iK and H, the real and the imaginary parts of the frequency q are given by

$$q_r = \frac{(\boldsymbol{\xi}, iK\boldsymbol{\xi})}{(\boldsymbol{\xi},\boldsymbol{\xi})}, \quad q_i = \left[-\frac{(\boldsymbol{\xi}, iK\boldsymbol{\xi})^2}{(\boldsymbol{\xi},\boldsymbol{\xi})^2} - \frac{(\boldsymbol{\xi}, H\boldsymbol{\xi})}{(\boldsymbol{\xi},\boldsymbol{\xi})}\right]^{\frac{1}{2}} \quad (78), (79)$$

respectively.

It follows from (66) and (78) that

$$|q_r| \leqslant \Omega_0 + (|m|+1)\,\Omega_{\max}, \quad (80)$$

where $\Omega_{\max}$ is the maximum of $|\Omega_1(\varpi)|$. Thus the azimuthal phase speed in the Ω_0 frame is bounded above by

$$|C_r| \leqslant \Omega_0 + 2\Omega_{\max}, \quad (81)$$

where we have defined $C_r + iC_i = C \equiv -q/m$.

Estimation of an upper bound on growth rates for the case of general differential rotation involves some difficulty as we are only able to give a bound which grows indefinitely as $|m| \to \infty$. However, it is possible to make an estimate in the case of a strong magnetic field when instability arises from the violation of (69) while (68) is satisfied. For this case, we have

$$|q_i|^2 \leqslant -(\boldsymbol{\xi}, H\boldsymbol{\xi})/(\boldsymbol{\xi},\boldsymbol{\xi}) \leqslant T_{\max}, \quad (82)$$

where $T_{\max} \equiv \max\limits_{\varpi}[-T(\varpi)]$ and T is defined by (69).

The case of uniform rotation is of particular interest and some concrete results are possible. Setting $\Omega_1 = 0$, it follows from (81) that the absolute value of the azimuthal phase speed is bounded above by Ω_0. Since all unstable modes drift westwards, we have

$$-\Omega_0 \leqslant C_r < 0. \quad (83)$$

For the growth rates, it follows from (77) and (79) that

$$q_i^2 \leqslant -(\boldsymbol{\xi}, H\boldsymbol{\xi})/(\boldsymbol{\xi},\boldsymbol{\xi}) \leqslant S_{\max}^2, \quad (84)$$

where $S_{\max}^2$ is the maximum of

$$S(\varpi)^2 \equiv -(N^2 - \varpi H_\varpi - 4\Omega_\phi^2). \quad (85)$$

It also follows from (78) and (79) that

$$|q|^2 = q_r^2 + q_i^2 = -(\boldsymbol{\xi}, H\boldsymbol{\xi})/(\boldsymbol{\xi},\boldsymbol{\xi}) \leqslant S_{\max}^2. \quad (86)$$

This upper bound remains unchanged if we replace q by q/m ($\equiv -c$). Thus

$$|c|^2 \leqslant S^2_{max}. \tag{87}$$

Before proceeding, we note that if $B_z = 0$ then a smaller upper bound is obtained, i.e.

$$|c|^2 \leqslant \max\,[-(N^2 - \varpi H_\varpi - 3\Omega^2_\phi)]. \tag{88}$$

Since the extension to include $B_z \neq 0$ is trivial for the discussion in this section, we shall make no distinction between the case $B_z = 0$ and the case $B_z \neq 0$ in the following discussion.

Acheson (1973) established a quadrant theorem, similar to Howard's (1961) semicircle theorem for plane parallel flow, for 'slow amplifying waves' when the rotation is uniform, the magnetic field purely toroidal and the Boussinesq approximation is made. 'Slow amplifying waves' are defined as the unstable modes occurring when the equilibrium configuration satisfies

$$\Omega^2_0 \geqslant N^2 + \Omega^2_\phi, \quad \text{so that} \quad |q|^2 \sim \Omega^2_\phi |N^2 + \Omega^2_\phi|/\Omega^2_0. \tag{89}$$

Then the quadrant theorem states that all such 'slow amplifying waves' lie within a quadrant with a radius of $\frac{1}{4}\Omega_0^{-1}\max\,(-N^2 + \varpi D\Omega^2_\phi)$ located in the second quadrant of the complex c plane.

According to the upper bounds estimated above [cf. (83), (84) or (87)], which are not restricted to 'slow amplifying modes' nor subject to the Boussinesq approximation, all unstable modes lie within a rectangle, with sides of length Ω_0 and S_{max}, located in the second quadrant of the complex c plane. The case $\Omega_0 \geqslant S_{max}$ is of particular interest for the problem of the earth's core. In this case, the upper bound on c^2_r is improved from a value of Ω^2_0 to S^2_{max} and we obtain a quadrant theorem valid for all unstable modes. This quadrant has a radius of S_{max}, which is larger by a factor of approximately $4\Omega_0 S^{-1}_{max}$ than the one valid for 'slow amplifying waves'.

5. Remarks on the normal-mode stability analysis

So far, we have assumed that the perturbation function has a normal-mode form

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\varpi; \phi, z)\exp(i\sigma t) \tag{90}$$

and said that the equilibrium state is stable if the imaginary part of all eigenvalues σ is $\geqslant 0$. This inference is strictly valid only if the eigenfunctions (i.e. perturbation functions) form a complete set. It is known that the set of eigenfunctions of the rotating MHD problem is not complete when perturbations are non-axisymmetric since then the singularity in the differential equation introduces the continuous spectrum. Therefore it is important that we perform a stability analysis of an initial-value problem without making the assumption (90).

Consider a linear vector equation

$$P\ddot{\boldsymbol{\xi}} + 2(iF_0 + F_1)\dot{\boldsymbol{\xi}} + Q\boldsymbol{\xi} = 0, \tag{91}$$

where a dot indicates $\partial/\partial t$ and the operators P, iF_0, F_1 and Q are Hermitian in an inner-product space defined by (33). Let

$$\alpha \equiv \inf_D \frac{(\boldsymbol{\eta}, P\boldsymbol{\eta})}{(\boldsymbol{\eta}, \boldsymbol{\eta})}, \quad \lambda \equiv \inf_D \frac{(\boldsymbol{\eta}, Q\boldsymbol{\eta})}{(\boldsymbol{\eta}, \boldsymbol{\eta})}, \tag{92}$$

where D is a set of admissible solutions of (91) satisfying some proper boundary conditions. We also define β_0 as the value of $(\dot{\xi}, P\dot{\xi})+(\xi, Q\xi)$ at the initial time $t = 0$. Then we have the following theorem.

Suppose that P, iF_0, F_1 and Q are Hermitian, with $P > 0$ and $F_1 \geqslant 0$. Then

(i) if $\lambda > 0$, $||\xi(t)|| \equiv (\xi, \xi)^{\frac{1}{2}}$ is bounded for all $t \geqslant 0$;

(ii) if $\lambda = 0$, $||\xi(t)|| \leqslant (\beta_0/\alpha)^{\frac{1}{2}} t + ||\xi(0)||$ for all $t \geqslant 0$.

This stability theorem is a straightforward extension of theorems proved by Barston (1970) for $F_0 \equiv 0$ and by Roberts & Soward (1972) for $F_1 \equiv 0$. The proof follows immediately by noting that

$$d[(\dot{\xi}, P\dot{\xi})+(\xi, Q\xi)]/dt = (P\ddot{\xi}+Q\xi, \dot{\xi})+(\dot{\xi}, P\ddot{\xi}+Q\xi)$$
$$= -4(\dot{\xi}, F_1\dot{\xi}) \leqslant 0, \tag{93}$$

which gives

$$(\dot{\xi}, P\dot{\xi})+(\xi, Q\xi) \leqslant \beta_0. \tag{94}$$

In the absence of, say, viscous dissipation, $F_1 \equiv 0$ in (91). Then the requirement that $(\xi, Q\xi)$ be positive definite gives a sufficient condition for the boundedness of $||\xi||$. Thus, in the normal-mode stability analysis, if the stability equation is formulated in the form (21) then the sufficient conditions for stability obtained by requiring positive-definiteness of $(\xi, Q\xi)$ remains valid (i.e. $||\xi||$ is bounded) even if the set of eigenfunctions is not complete. For this reason, the sufficient conditions for stability such as (45), (58) [see the comment following (58)], (63) and (68) and (69) remain valid. On the other hand, the validity of sufficient conditions for stability such as (55) and (74) remains open.

If the rotation is uniform, then the stability equation in the presence of viscous dissipation may be written in the form (91) with all the hypotheses of the theorem satisfied. Thus the sufficient conditions for stability (58) remain valid in the presence of viscosity. However, a similar conclusion has not been reached in the case of differential rotation, since it is not possible to write the linear stability equation in the form (91) with all the hypotheses of the theorem satisfied.

6. Concluding remarks

As pointed out by a referee, there is the question of whether it is possible for sufficient conditions for stability such as (58) to be satisfied near the axis as $\varpi \to 0$ when the inner wall is removed and the axial magnetic field neglected. We shall consider this important question in the non-rotating case.

When $\Omega = 0 = B_z$, an application of Barston's (1970) theorem gives as the necessary and sufficient conditions for stability

$$-g_\varpi D\rho - \varpi^{-2} D(\varpi B_\phi^2) \geqslant 0 \quad \text{everywhere}. \tag{95}$$

Here we have set $|m| = 1$ since these perturbations are the least stable non-axisymmetric ones. We note that, if we put $B_z = 0$ in (58) and then multiply this equation by ρ, it reduces to (95). Near the axis, we have the following approximations:

$$g_\varpi \sim g_0 \varpi, \tag{96}$$

$$\rho \sim \rho_0 - \rho_\alpha \varpi^\alpha, \quad p \sim p_0 - p_\beta \varpi^\beta, \quad B_\phi \sim B_0 \varpi^\gamma, \tag{97}$$

where g_0, ρ_0, ρ_α, p_0, p_β, B_0, α, β and γ are positive constants. Approximation (96) is a consequence of Poisson's equation for self-gravitation while the minus sign in front of ρ_α and p_β is a consequence of the assumption that both density and pressure fall off away from the axis.

Using (96) and (97), the equilibrium condition (5) with $\Omega = 0 = B_z$ can be written as

$$\rho_0 g_0 \varpi \sim \beta p_\beta \varpi^{\beta-1} - (\gamma+1) B_0^2 \varpi^{2\gamma-1}. \tag{98}$$

Clearly, we must have $\beta \geqslant 2$ and $\gamma \geqslant 1$ in order to satisfy (98). The latter requirement shows that B_ϕ cannot be singular as $\varpi \rightarrow 0$. The restriction on α is that $\alpha \geqslant 0$. Using these properties, (95) has the following approximate form near the axis:

$$\alpha \rho_\alpha g_0 \varpi^\alpha \sim (2\gamma+1) B_0^2 \varpi^{2(\gamma-1)} \geqslant 0. \tag{99}$$

For a homogeneous fluid, $\alpha \equiv 0$ and (99) is always violated. Thus instability must occur near the axis. To prevent such instability, the distribution of density and toroidal field must satisfy the condition

$$2(\gamma-1) > \alpha \quad \text{or} \quad \gamma > 1 + \tfrac{1}{2}\alpha. \tag{100}$$

For example, if $\alpha = 1{\cdot}5$ and $\gamma = 2$, then instability near the axis does not occur.

The occurrence of instability near the axis may be explained as the competition between the stabilizing influence of the gravitational buoyant force and the destabilizing influence of the magnetic buoyant force. It is known that the magnetic buoyant force may be destabilizing or stabilizing depending on how the magnetic field is stratified. For the region away from the axis, a distribution such as $B_\phi \propto \varpi^{-1}$ is allowed and the magnetic buoyant force is stabilizing. But such a distribution is not permitted near the axis, where the equilibrium condition requires that $B_\phi \propto \varpi^\gamma$ with $\gamma \geqslant 1$, and the magnetic buoyant force is always destabilizing in this region. This explains why instability near the axis must occur in a homogeneous fluid since there is no stabilizing gravitational buoyant force to suppress it. For the same reason, such instability must occur when the density distribution is unstable ($D\rho > 0$) near the axis.

If the equilibrium configuration is spheroidal, then the situation is less optimistic. Following the argument of the competing effects of the buoyant forces, it is easy to see that instability near the axis may be suppressed at least in the neighbourhood of the core if the density distribution is stable there and (100) is satisfied. However, the ϖ component of the gravitational buoyant force becomes negligible near the north and south poles of the spheroid since the ϖ component of the gravitational acceleration is very small in these two regions. This means that the stabilizing influence of the gravitational buoyant force in the ϖ direction in these two regions is too weak to overcome the destabilizing influence of the magnetic buoyant force in that direction. Thus instability near the axis may still occur in the neighbourhoods of the south and the north poles. To suppress it, there are two possibilities: one is by rotation and the other by a poloidal magnetic field. We hope to investigate this problem in the future.

I wish to thank the referees for their very constructive suggestions. The financial support of the Science Research Council is gratefully acknowledged.

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